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Ideals and Filters in Pseudo-Effect Algebras

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In this paper, we show that the filters and local filters are equivalent in pseudo-effect algebras. Ideals and local ideals and generalized ideals are equivalent in the pseudo-effect algebras, too.

KEY WORDS: pseudo-effect algebras; ideals; filters; local ideals; local filters; generalized ideals.

1. INTRODUCTION

Foulis and Bennet in 1994 introduced the following algebraic system $(E, \perp, \oplus, 0, 1)$ to model unsharp quantum logics, and $(E, \perp, \oplus, 0, 1)$ is said to be an *effect algebra* (Foulis and Bennet, 1994):

Let E be a set with two special elements 0, 1, $\perp$ be a subset of $E \times E$, if $(a, b) \in \perp$, denote $a \perp b$, and let $\oplus : \perp \to E$ be a binary operation, and the following axioms hold:

(E1) (Commutative Law) If $a, b \in E$ and $a \perp b$, then $b \perp a$ and $a \oplus b = b \oplus a$.

(E2) (Associative Law) If $a, b, c \in E$, $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$ and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

(E3) (Orthocomplementation Law) For each $a \in E$ there exists a unique $b \in E$ such that $a \perp b$ and $a \oplus b = 1$.

(E4) (Zero-Unit Law) If $a \in E$ and $1 \perp a$, then $a = 0$.

An *orthoalgebra* is an effect algebra in which the Zero-Unit Law is replaced by the stronger (Foulis *et al.*, 1992):

(E5) (Consistency Law) If $a \in E$ and $a \perp a$, then $a = 0$.

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Dvurecenskij and Vetterleinthat in 2001 dropped the Commutative Law of effect algebras and introduced a new quantum logic structure and called it the *pseudo-effect algebra* (Dvurecenskij and Vetterleinthat, 2001):

Let PE be a set with two special elements 0, 1, $\perp$ be a subset of $PE \times PE$, if $(a, b) \in \perp$, denote $a \perp b$, and let $\oplus : \perp \to PE$ be a binary operation, and the following axioms hold:

(PE1) $a \oplus b$, $(a \oplus b) \oplus c$ exist iff $b \oplus c$, $a \oplus (b \oplus c)$ exist, and in this case, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(PE2) For each $a \in PE$, there is exactly one $d \in PE$, and exactly one $e \in PE$ such that $a \oplus d = e \oplus a = 1$.
(PE3) If $a \oplus b$ exists, there are elements $d, e \in PE$ such that $a \oplus b = d \oplus a = b \oplus e$.
(PE4) If $1 \oplus a$ or $a \oplus 1$ exist, then $a = 0$.

In view of (PE2), we may define the two unary operation $\sim$ and $-$ by requiring for any $a \in PE$,

$$a \oplus a^{\sim} = a^{-} \oplus a = 1.$$

Lemma 1.1. (Dvurecenskij and Vetterleinthat, 2001). *Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra. For $a, b, c \in PE$, we have*

(i) $a \oplus 0 = 0 \oplus a = a$.
(ii) $a \oplus b = 0$ *implies that* $a = b = 0$.
(iii) $0^{\sim} = 0^{-} = 1$, $1^{\sim} = 1^{-} = 0$.
(iv) $a^{\sim -} = a^{- \sim} = a$.
(v) $a \oplus b = a \oplus c$ *implies* $b = c$, *and* $b \oplus a = c \oplus a$ *implies* $b = c$ *(cancellation laws).*
(vi) $a \oplus c = b$ *iff* $a = (c \oplus b^{\sim})^{-}$ *iff* $c = (b^{-} \oplus a)^{\sim}$.
(vii) $a \oplus b$ *exists iff* $a \leq b^{-}$ *iff* $b \leq a^{\sim}$.

Moreover, we can define a partial order for pseudo-effect algebras, that is, $a \leq b$ iff there exists some $c \in PE$, $a \perp c$ and $a \oplus c = b$.

It follows from (PE3) that $a \leq b$ iff there exists some $d \in PE$, $d \perp a$ and $d \oplus a = b$.

Recently, Ma, Wu, and Lu (2004) introduced two partial operations $\ominus_l$ and $\ominus_r$ in pseudo-effect algebras as following:

Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra, $a, b, c \in PE$. If $a \leq b$ and $c \oplus a = b$, we define c as the *left difference* of b and a, and denote $c = b \ominus_l a$. Dually, if $a \leq b$ and $a \oplus d = b$, we define d as the *right difference* of b and a, and denote $d = b \ominus_r a$.

It follows from (v) and (vi) of Lemma 1.1 that the two operations $\ominus_l$ and $\ominus_r$ are well defined, and if $a \leq b$, then $(b \ominus_l a) = (a \oplus b^{\sim})^{-}$, $(b \ominus_r a) = (b^{-} \oplus a)^{\sim}$.

Lemma 1.2. (Ma, Wu, and Lu, 2004). *Let* $(PE, \oplus, \perp, 0, 1)$ *be a pseudo-effect algebra,* $a \leq b \leq c$. *Then we have*

(PD1) $b \ominus_l a \leq b, b \ominus_r a \leq b$.
(PD2) $b \ominus_l (b \ominus_r a) = a, b \ominus_r (b \ominus_l a) = a$.
(PD3) $(c \ominus_l b) \leq (c \ominus_l a), (c \ominus_r b) \leq (c \ominus_r a)$.
(PD4) $(c \ominus_l a) \ominus_r (c \ominus_l b) = b \ominus_l a, (c \ominus_r a) \ominus_l (c \ominus_r b) = b \ominus_r a$.
(PD5) *If* $1 \ominus_r (1 \ominus_l b \ominus_l a)$ *is defined, then there exist* $d, e \in PE$ *such that*

$$(1 \ominus_r (1 \ominus_l b \ominus_l a)) = (1 \ominus_r (1 \ominus_l a \ominus_l d)) = (1 \ominus_r (1 \ominus_l e \ominus_l b)).$$

If $1 \ominus_l (1 \ominus_r b \ominus_r a)$ *is defined, then there exists* $f, g \in PE$ *such that*

$$(1 \ominus_l (1 \ominus_r b \ominus_r a)) = (1 \ominus_l (1 \ominus_r a \ominus_r f)) = (1 \ominus_l (1 \ominus_r g \ominus_r b)).$$

Lemma 1.3. (Ma, Wu, and Lu, 2004). *Let* $(PD, \leq, 0, 1)$ *be a pseudo-effect algebra. Then we have*

(PD6) $c \ominus_l a \ominus_r b = c \ominus_r b \ominus_l a$.
(PD7) $(c \ominus_l a) \ominus_l (b \ominus_l a) = (c \ominus_l b), (c \ominus_r a) \ominus_r (b \ominus_r a) = (c \ominus_r b)$.

Lemma 1.4. (Ma, Wu, and Lu, 2004). *Let* $(PE, \oplus, \perp, 0, 1)$ *be a pseudo-effect algebra. Then* $(1 \ominus_l (1 \ominus_r a \ominus_r b))$ *exists iff* $(1 \ominus_r (1 \ominus_l b \ominus_l a))$ *exists iff* $a \oplus b$ *exists and*

$$a \oplus b = (1 \ominus_l (1 \ominus_r a \ominus_r b)) = (1 \ominus_r (1 \ominus_l b \ominus_l a)).$$

Lemma 1.5. *Let* $(PE, \oplus, \perp, 0, 1)$ *be a pseudo-effect algebra and* $b \leq a$. *Then*

$$(1 \ominus_l a) \oplus b = 1 \ominus_l (a \ominus_r b),$$

$$b \oplus (1 \ominus_r a) = 1 \ominus_r (a \ominus_l b).$$

Proof: From $1 \ominus_r (1 \ominus_l a) = a$ and Lemma 1.4, we have

$$(1 \ominus_l a) \oplus b = 1 \ominus_l (1 \ominus_r (1 \ominus_l a) \ominus_r b) = 1 \ominus_l (a \ominus_r b).$$

Similarly, we can prove that

$$b \oplus (1 \ominus_r a) = 1 \ominus_r (a \ominus_l b). \qquad \square$$

2. IDEALS AND FILTERS

As we knew, the study of the algebra properties of quantum logic structures is a very important project (Miklos, 1998), we also knew that the ideals and filters of quantum logic structures are powerful notions. Recently, Jing showed that the ideals and local ideals are equivalent in effect algebras, the filters and local filters are also equivalent in effect algebras, moreover, he proved also that each ideal in pseudo-effect algebra must be a local ideal, each filter in pseudo-effect algebra must be also a local filter (Jing, 2003). Shang and Li introduced the generalized ideals in orthoalgebras and proved that the generalized ideals and the local ideals are equivalent in the orthoalgebras.

In this paper, we use the new methods, that is, by means of the two operations $\ominus_l$ and $\ominus_r$ which were introduced in (Ma, Wu, and Lu, 2004), we show that ideals and local ideals are equivalent in pseudo-effect algebras, filters and local filters are also equivalent in pseudo-effect algebras, moreover, ideals and generalized ideals are equivalent in pseudo-effect algebras, too.

Definition 2.1. (Jing, 2003). Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra, I be a nonempty subset of $(PE, \oplus, \perp, 0, 1)$. If

(I1) $x \in I$, $y \in PE$, $y \leq x$ implies $y \in I$.
(I2) $x \ominus_l y \in I$, $y \in I$ imply $x \in I$, $x \ominus_r y \in I$, $y \in I$ imply $x \in I$.

Then I is said to be an *ideal* of $(PE, \oplus, \perp, 0, 1)$.

Definition 2.2. (Jing, 2003). Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra, F be a nonempty subset of $(PE, \oplus, \perp, 0, 1)$. If

(F1) $a \in F$, $b \in PE$, $a \leq b$ implies $b \in F$.
(F2) $a \in F$, $b \in PE$, $b \leq a$, and either $(1 \ominus_l a) \oplus b \in F$ or $b \oplus (1 \ominus_r a) \in F$ implies $b \in F$.

Then F is said to be a *filter* of $(PE, \oplus, \perp, 0, 1)$.

Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra. A triple $\{p, q, r\} \in PE$ is said to be a *left triangle* if $r \oplus p$, $r \oplus q$, $(r \oplus p) \oplus q$, and $(r \oplus q) \oplus p$ exist in PE, and is denoted by $\triangle(r \triangleright p, q)$.

Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra. A triple $\{p, q, r\} \in PE$ is said to be a *right triangle* if $p \oplus r$, $q \oplus r$, $p \oplus (q \oplus r)$, and $q \oplus (p \oplus r)$ exist in PE, and is denoted by $\triangle(p, q \Delta r)$.

Definition 2.3. (Jing, 2003). Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra, F be a nonempty subset of $(PE, \oplus, \perp, 0, 1)$. If for every right triangle $\triangle(p, q \Delta r) \in PE$,

$$(p \oplus r), (q \oplus r) \in F \Leftrightarrow r \in F.$$

Then F is said to be a *local filter* of PE.

Dually, we can define

Definition 2.4. (Jing, 2003). Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra, F be a nonempty subset of $(PE, \oplus, \perp, 0, 1)$. If for every left triangle $\triangle(r \Delta p, q) \in PE$,

$$(r \oplus p), (r \oplus q) \in F \Leftrightarrow r \in F.$$

Then F is said to be a local filter of PE.

Jing showed that Definition 3 and Definition 3' are equivalent (Jing, 2003).

Definition 2.5. (Jing, 2003). Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra, I be a nonempty subset of $(PE, \oplus, \perp, 0, 1)$. Then I is said to be a *local ideal* if $I^{-} = \{1 \ominus_l p : p \in I\}$ (equivalently, $I^{\sim} = \{1 \ominus_r p : p \in I\}$) is a local filter.

Definition 2.6. (Shang and Li, 2003). Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra, I be a nonempty subset of $(PE, \oplus, \perp, 0, 1)$. Then I is said to be a *generalized ideal* if

(GI1) $x \in I, y \in PE, y \leq x$ implies $y \in I$.
(GI2) Let $\{p, q, r\} \subseteq PE$ be a right triangle. If $p \oplus r \in I, q \oplus r \in I$, then $p \oplus q \oplus r \in I$.

It follows easily from the definition of local ideals that

Lemma 2.6. *Let $(PE, \oplus, \perp, 0, 1)$ be a pseudo-effect algebra. Then a nonempty subset I of PE is a local ideal iff for every right triangle $\triangle(p, q\Delta r) \in PE$,*

$$1 \ominus_l (p \oplus r), 1 \ominus_l (q \oplus r) \in I \Leftrightarrow (1 \ominus_l r) \in I,$$

iff for every right triangle $\triangle(p, q\ (r) \in PE$,

$$1 \ominus_r (p \oplus r), 1 \ominus_r (q \oplus r) \in I \Leftrightarrow (1 \ominus_r r) \in I,$$

iff for every left triangle $\triangle(r\Delta p, q) \in PE$,

$$1 \ominus_l (r \oplus p), 1 \ominus_l (r \oplus q) \in I \Leftrightarrow (1 \ominus_l r) \in I,$$

iff for every left triangle $\triangle(r\Delta p, q) \in PE$,

$$1 \ominus_r (r \oplus p), 1 \ominus_r (r \oplus q) \in I \Leftrightarrow (1 \ominus_r r) \in I.$$

3. MAIN RESULTS

Now, we prove the main theorems of this paper.

Theorem 3.1. *Let* (PE, $\oplus, \perp, 0, 1)$ *be a pseudo-effect algebra. If* F *is a local filter of PE, then F is also a filter of* PE.

Proof: Let F be a local filter of PE. If $a \in F, b \in PE, a \leq b$, let $p = b \ominus_l a, q = 0, r = a$, then $\{p, q, r\}$ is a right triangle, note that $r = a \in F$, it follows from the definition of local filter that $(b \ominus_l a) \oplus a = b \in F$. Thus, the condition (F1) is proved.

Now assume that $a \in F, b \in PE, b \leq a, (1 \ominus_l a) \oplus b \in F$. Let $r = b$, $p = (a \ominus_l b)$, $q = (1 \ominus_l a)$. It is easy to prove that $\{a \ominus_l b, (1 \ominus_l a), b\}$ is a right triangle. Note that $p \oplus r = a \in F, q \oplus r = (1 \ominus_l a) \oplus b \in F$, so by the definition of local filter $b = r \in F$.

Similarly, we may prove that if $a \in F, b \in PE, b \leq a$, and $b \oplus (1 \ominus_r a) \in F$, then $b \in F$.

Thus, the condition (F2) is proved, so the theorem holds.

It follows from Theorem 1 and the definition of local ideals that if I is a local ideal of $(PE, \oplus, \perp, 0, 1)$, then $0 \in I$. □

Theorem 3.2. *Let* (PE, $\oplus$, $\perp$, 0, 1) *be a pseudo-effect algebra. If I is a local ideal of* PE*, then I is an ideal of PE.*

Proof: Let I be a local ideal of PE. If $a \in I, b \in PE, a \leq b$, let $r = (1 \ominus_r b)$, $p = (b \ominus_r a)$, $q = b$, then it follows from Lemma 1.4 that

$$p \oplus r = 1 \ominus_r [1 \ominus_l (1 \ominus_r b) \ominus_l (b \ominus_r a)] = 1 \ominus_r [b \ominus_l (b \ominus_r a)] = 1 \ominus_r a,$$

and

$$q \oplus r = 1.$$

Since $1 \ominus_l (p \oplus r) = a \in I, 1 \ominus (q \oplus r) = 1 \ominus_l 1 = 0 \in I$, it follows from Lemma 2.6 that $1 \ominus_l r = b \in I$, so the condition (I1) is proved.

If $a \in I, b \in PE, a \leq b, b \ominus_r a \in I$, denote $r = 1 \ominus_l b$, $p = b \ominus_l a$, $q = a$, it follows from Lemma 1.5 that $r \oplus q = (1 \ominus_l b) \oplus a = 1 \ominus_l (b \ominus_r a)$, $r \oplus p = (1 \ominus_l b) \oplus (b \ominus_l a) = 1 \ominus_l [b \ominus_r (b \ominus_l a)] = 1 \ominus_l a$. Note that

$$1 \ominus_r (r \oplus q) = b \ominus_r a \in I, 1 \ominus_r (r \oplus p) = a \in I.$$

It follows from Lemma 2.6 again that

$$1 \ominus_r r = 1 \ominus_r (1 \ominus_l b) = b \in I.$$

Similar, we may prove that if $a \in I, b \in PE, a \leq b, b \ominus_l a \in I$, then $b \in I$. Thus, the condition (I2) is proved, so the theorem holds. □

Theorem 3.3 *Let* (PE, $\oplus$, $\perp$, 0, 1) *be a pseudo-effect algebra. Then I is an ideal of* PE *iff I is a generalized ideal of* PE.

Proof: If I is a generalized ideal and $a \in I, a \leq b, b \ominus_l a \in I$. Let $p = a, q = b \ominus_l a, r = 0$, then $p \oplus r \in I, q \oplus r \in I$, it follows from the definition of generalized ideals that $(p \oplus q) \oplus r = (b \ominus_l a) \oplus a \oplus 0 = b \in I$. Similarly, if $a \in I, a \leq b, b \ominus_r a \in I$ we may prove that $b \in I$, too. Thus, the condition (I2) holds and so I is an ideal.

If I is an ideal. Let $\{p, q, r\} \subseteq PE$ be a right triangle and $p \oplus r \in I, q \oplus r \in I$. Note that $p \leq p \oplus r$ and the condition (I1) that $p \in I$. On the other hand, since

$$(p \oplus q \oplus r) \ominus_l (q \oplus r) = p \in I.$$

So it follows from the condition (I2) that $(p \oplus q \oplus r) \in I$. Thus, we proved that I is a generalized ideal. The theorem is proved. □

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